B.Sc Part I (Hons)

Paper II (Partial Differentiation with ...)

**Partial Differentiation:–** Let $u = f(x, y)$ be a function of two variables $x$ and $y$.

The partial derivative of $u$ with respect to $x$ is denoted by the symbol $\frac{\partial u}{\partial x}$ and this means that $u$ has been differentiated w.r.to $x$ only the remaining variable $y$ has been treated as constant.

i.e. if $u = f(x, y)$, then

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

if this limit exists.

Similarly, the partial derivative of $u$ w.r.to $y$ is denoted by $\frac{\partial u}{\partial y}$ and this means that $u$ has been differentiated with respect to $y$ only and the remaining variable i.e. if $u = f(x, y)$, then

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$ if this limit exists.

**Example 1.** Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ when $u = \log(x^3+y^3+z^3-3xyz)$

**Solution:–** According to the rule of differentiating a function of function, we have

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)$$

$$= \frac{3x^2+0+0-3yz}{x^3+y^3+z^3-3xyz}$$ (Since $y$ and $z$ have been treated as constants)

$$= \frac{3(y^2-yz)}{x^3+y^3+z^3-3xyz}$$

Similarly $$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz} \times \frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$

$$= \frac{0+3y^2+0-3xz}{x^3+y^3+z^3-3xyz}$$ (Here $x$ and $z$ have been treated as constants)

$$= \frac{3(y^2 - xz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \times \frac{\partial}{\partial x}\left(x^3+y^3+z^3-3xyz\right)$$

$$= \frac{0+0+3x^2-3xy}{x^3+y^3+z^3-3xyz}$$

Here $x$ and $y$ have been treated as Contains.

$$= \frac{3(x^2-xy)}{x^3+y^3+z^3-3xyz}$$

Example 2 If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Solution:_ Give that $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ ——(1)

Therefore differentiating partially with respect to $x$,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{(1-(x/y)^2)}} \cdot \frac{1}{y} + \frac{1}{1+(y/x)^2} \cdot y\left(-\frac{1}{x^2}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} - \frac{x^2}{x^2+y^2} \cdot \frac{y}{x^2}$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2} \quad \therefore x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \text{ ——(2)}$$

Again differentiating (1) Partially w.r.to $y$, we have

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{[1-(x/y)^2]}} \cdot x\left(-\frac{1}{y^2}\right) + \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$= -\frac{y}{\sqrt{y^2-x^2}}\cdot\frac{x}{y^2} + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\therefore y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \text{ ——(3)}$$

Now adding (2) and (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0. \text{ Proved.}$$

Paper II (B.Sc Part I)

Examples of Partial differentiation.

Example(4): If $\frac{1}{u} = \sqrt{x^2+y^2+z^2}$, show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + u = 0$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Soln. Given $\frac{1}{u} = (x^2+y^2+z^2)^{\frac{1}{2}}$

$\therefore \frac{1}{u^2} = x^2+y^2+z^2$

Hence differentiating partially with respect to $x$, we get

$-\frac{2}{u^3}\frac{\partial u}{\partial x} = 2x \Rightarrow \frac{\partial u}{\partial x} = -xu^3 \Rightarrow x\frac{\partial u}{\partial x} = -x^2u^3$ ——— (1)

Similarly $y\frac{\partial u}{\partial y} = -u^3y^2$ ——— (2)

$z\frac{\partial u}{\partial z} = -u^3z^2$ ——— (3)

Adding (1), (2) and (3), we get

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u^3(x^2+y^2+z^2)$

$= -u^3 \cdot \frac{1}{u^2} = -u.$

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + u = 0$ Proved

Again for the second prove, we see that

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-u^3x\right)$

$= -\left(u^3 \cdot 1 + x \cdot 3u^2\frac{\partial u}{\partial x}\right) = -\left(u^3 + 3xu^2(-u^3x)\right)$

$= -u^3 + 3x^2u^5$ ——— (4)

Similarly $\frac{\partial^2 u}{\partial y^2} = -u^3 + 3y^2u^5$ ——— (5)

$\frac{\partial^2 u}{\partial z^2} = -u^3 + 3z^2u^5$ ——— (6)

Adding (4), (5) and (6), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3u^3 + 3u^5(x^2+y^2+z^2)$$

$$= -3u^3 + 3u^5 \cdot \frac{1}{u^2} = \cancel{-3u^3} + \cancel{3u^3} = 0$$

Proved

Example (8): - If $u = \log(x^3+y^3+z^3-3xyz)$, show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -9/(x+y+z)^2$

Soln. — Given that $u = \log(x^3+y^3+z^3-3xyz)$

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3yz / x^3+y^3+z^3-3xyz$

$\frac{\partial u}{\partial y} = 3y^2 - 3zx / x^3+y^3+z^3-3xyz$

$\frac{\partial u}{\partial z} = 3z^2 - 3xy / x^3+y^3+z^3-3xyz$

Adding, we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$

$= 3(x^2+y^2+z^2-xy-yz-zx) / (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$= 3/x+y+z$ Proved.

(ii) We have

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= 3\left\{\frac{-1}{(x+y+z)^2}\right\} + 3\left\{\frac{-1}{(x+y+z)^2}\right\} + 3\left\{\frac{-1}{(x+y+z)^2}\right\}$$

$$= \frac{-9}{(x+y+z)^2}$$ Proved.

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